Large Forbidden Configurations and Design Theory

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Abstract

Let forb(m, F) denote the maximum number of columns possible in a (0,1)-matrix A that has no repeated columns and has no submatrix which is a row and column permutation of F. We consider cases where the configuration F has a number of columns that grows with m. For a $k \times \ell$ matrix G, define $s \cdot G$ to be the concatenation of s copies of G. In a number of cases we determine forb$(m, m^\alpha \cdot G)$ is $\Theta(m^{k+\alpha})$. Results of Keevash on the existence of designs provide constructions that provide asymptotic lower bounds. An induction idea of Anstee and Lu is useful in obtaining upper bounds.

Keywords: extremal set theory, BIBD, t-designs, (0,1)-matrices, multiset, forbidden configurations, trace, subhypergraph.

1 Introduction

We first give the matrix notation for our extremal problems. Define a matrix to be *simple* if it is a (0,1)-matrix with no repeated columns. Define that F is a *configuration* in A (denoted $F \prec A$) if there is a submatrix of A which is a row and column permutation of

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F. If we think of a simple matrix as the element-set incidence matrix of a set system, then a configuration corresponds to the *trace*. Define:

$$\text{Avoid}(m, F) = \{A \,:\, A \text{ is } m\text{-rowed simple}, F \not\prec A\},$$
$$\text{forb}(m, F) = \max_A\{\|A\| : A \in \text{Avoid}(m, F)\}.$$

For two matrices X, Y on the same number of rows let $[X|Y]$ denote the concatenation of X and Y. Let $s \cdot F = [F|F|\cdots|F]$ be the concatenation of s copies of F. Considering s to be a growing function of m and computing $\text{forb}(m, s\cdot F)$ is the focus of this paper. In this paper we typically choose $s = m^\alpha$. Think of m^α as an 'honourary integer'. Of course it would be correct to write $\lfloor m^\alpha \rfloor$ instead but this is not done to keep the presentation simpler. Let K_k denote the $k \times 2^k$ matrix of all $(0,1)$-columns on k rows and let K_k^ℓ denote the $k \times \binom{k}{l}$ matrix of all possible $(0,1)$-columns of column sum ℓ on k rows. Let $\mathbf{1}_k$ (respectively $\mathbf{0}_k$) denote the $k \times 1$ columns of 1's (resp. 0's).

Computing $\text{forb}(m, s \cdot \mathbf{1}_k)$ requires some results. Recent breakthrough results of Peter Keevash give the existence of simple designs that are useful to provide lower bounds. Given parameters t, m, k, λ, a t-(m, k, λ) *design* $\mathcal{D}$ is a multiset of subsets in $\binom{[m]}{k}$ such that for each $S \in \binom{[m]}{t}$ there are exactly λ blocks $B \in \mathcal{D}$ containing S. A t-(m, k, λ) design $\mathcal{D}$ is *simple* if $\mathcal{D}$ is a set (i.e. no repeated blocks). A t-(m, k, λ) design $\mathcal{D}$ can be encoded as a $m \times \frac{\lambda}{\binom{k}{t}}\binom{m}{t}$ element-block incidence matrix A. Then A is simple if and only if the design is simple. Then each t-tuple of rows contains $\lambda \cdot \mathbf{1}_t$ and $A \in \text{Avoid}(m, (\lambda+1)\cdot \mathbf{1}_t)$. A quite complete result for simple triple systems had already been given.

Theorem 1.1 *(Dehon [10]) Let m, λ be given. Assume $m \geq \lambda+2$ and $m \equiv 1, 3 (mod\ 6)$. Then there exists a simple $2-(m, 3, \lambda)$ design.*

Corollary 1.2 *Let λ be given with $\lambda \leq m-2$. Assume $m \equiv 1, 3(mod\ 6)$. Then $forb(m, (\lambda+2)\cdot \mathbf{1}_2) = \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \frac{\lambda}{3}\binom{m}{2}$.*

Proof: Let M be the element-block incidence matrix of a simple $2-(m, 3, \lambda)$ design. Then $M \in \text{Avoid}(m, (\lambda+1)\cdot \mathbf{1}_2)$. Let $B = [K_m^0|K_m^1|K_m^2|M]$. Then $B \in \text{Avoid}(m, (\lambda+2)\cdot \mathbf{1}_2)$. ■

Corollary 1.3 *Let $0 < \alpha \leq 1$ be given. Then $forb(m, m^\alpha \cdot \mathbf{1}_2)$ is $\Theta(m^{2+\alpha})$.*

Proof: Let $\lambda = m^\alpha - 2$. We apply the construction of Conjecture 1.2 when the divisibility conditions for m are satisfied. When m does not satisfy the divisibilty conditions then choose the largest $m' < m$ that does satisfy divisibility conditions. Again find M as the $m' \times \frac{m^\alpha - 2}{3}\binom{m'}{2}$ element-block incidence matrix of a $2-(m, 3, \lambda)$ design. Let $B = [K_{m'}^0|K_{m'}^1|K_{m'}^2|M]$. Then $B \in \text{Avoid}(m', (\lambda+2)\cdot \mathbf{1}_2)$. Now add $m - m'$ rows of 1's to finish the construction of a matrix $A \in \text{Avoid}(m, m^\alpha)$ with $\|A\|$ being

$\binom{m'}{0}+\binom{m'}{1}+\binom{m'}{2}+\frac{m^\alpha-2}{3}\binom{m'}{2}$ This is $\Theta((m')^2m^\alpha)$ and since $m'>m/c$ for some constant $c>0$ (we need only a weak estimate) we have that $\|A\|$ is $\Theta(m^{2+\alpha})$. ■

The following remarkable existence theorem appears as Theorem 6.6 in [12]. Note that the divisibility conditions are what is required for a t-$(m,k,1)$ simple design. Keevash has a slightly different theorem for larger λ which would have slightly weaker divisibility conditions. A surprising part of this result is that the design is simple and hence can be used as a construction for our problems.

Theorem 1.4 *(Keevash [12]) Let $1/m \ll \theta \ll 1/k \le 1/(t+1)$ and $\theta \ll 1$. Suppose that $\binom{k-i}{t-i}$ divides $\binom{m-i}{t-i}$ for $0\le i\le r-1$. Then there exists a t-(m,k,λ) simple design for $\lambda \le \theta m^{k-t}$.*

Note that $1\le\lambda\le\binom{m}{k}\binom{k}{t}/\binom{m}{t}$ and so this covers a fraction θ of the possible range for λ. Baranyai's Theorem when applied to decompose the complete r-uniform hypergraph into sets of disjoint edges, each set of size m/r, yields the following result that is helpful for $t=1$.

Theorem 1.5 *[9] Let m,r be given with r divides m. Then we can create $\binom{m-1}{r-1}$ matrices $M_1,M_2,\ldots,M_{\binom{m-1}{r-1}}$ such that $K_m^r=[M_1|M_2|\cdots|M_{\binom{m-1}{r-1}}]$ where each M_i consists of m/r columns of sum r and each row sum is 1.*

A warmup is the following exact result.

Proposition 1.6 *Let $t\ge1$ be given. Then forb$(m,\left(1+\sum_{i=t}^{r}\binom{m-t}{i-t}\right)\cdot\mathbf{1}_t)=\sum_{i=0}^{r}\binom{m}{i}$.*

Proof: We have the construction $[K_m^0K_m^1\cdots K_m^r]$ which has no configuration $(m+\sum_{i=t}^{r}\binom{m-2}{i-t}))\cdot\mathbf{1}_t$. The upper bound follows by the pigeonhole bound (1). We would preferentially choose the columns of smallest column sum. It is easy to verify that K_m^i contains $\binom{m-t}{i-t}\cdot\mathbf{1}_t$ in each i-tuple of rows but $(1+\binom{m-t}{i-t}))\cdot\mathbf{1}_t\not\prec K_m^i$. ■

Given parameters t,m,k,λ, define a t-(m,k,λ) *packing* $\mathcal{P}$ to be a set of subsets in $\binom{[m]}{k}$ such that for each $S\in\binom{[m]}{t}$ there are at most λ blocks $B\in\mathcal{D}$ containing S. For this paper repeated blocks are not allowed. Such a packing $\mathcal{P}$, when encoded by its element-block incidence matrix, yields a $m\times|\mathcal{P}|$ matrix in Avoid$(m,(\lambda+1)\cdot\mathbf{1}_t)$.

Theorem 1.7 *Let $t\ge1$ be given. There is a $t-(m,k,m^\alpha)$ packing $\mathcal{P}$ with $|\mathcal{P}|$ being $\Theta(m^{t+\alpha})$ and so forb$(m,m^\alpha\cdot\mathbf{1}_t)$ is $\Theta(m^{t+\alpha})$.*

Proof of Theorem 1.7: Let $A\in$ Avoid$(m,s\cdot\mathbf{1}_t)$. Let a_i be the number of columns of sum i. The pigeonhole bound becomes

$$\|A\|\le\sum_{i=t}^{m}a_i\binom{i}{t}\le(s-1)\binom{m}{t}\qquad(1)$$

Of course we also have $a_i \le \binom{m}{i}$.

First consider $t=1$ and apply Baranyai's Theorem 1.5. Determine a value r so that

$$\left(1+\sum_{i=t}^{r}\binom{m-1}{i-1}\right) < m^\alpha < \left(1+\sum_{i=t}^{r+1}\binom{m-1}{i-1}\right).$$

Assume r divides m. Then let $v = m^\alpha - \left(1+\sum_{i=t}^{r}\binom{m-1}{i-1}\right)$ and use Theorem 1.5 to find v 'perfect matchings', namely $M_1, M_2, \ldots, M_v$ and then $[K_m^0 K_m^1 \cdots K_m^r M_0 M_1 \cdots M_v] \in \mathrm{Avoid}(m, m^\alpha \cdot \mathbf{1}_t)$ with $\Theta(m^{t+\alpha})$ columns. The asymptotics follow for cases where r does not divide m.

For $t \ge 2$ follow a similar argument but now using Theorem 1.4. Determine a value r with

$$\left(1+\sum_{i=t}^{r}\binom{m-t}{i-t}\right) < m^\alpha < \left(1+\sum_{i=t}^{r+1}\binom{m-t}{i-t}\right).$$

Let $s = m^\alpha - \left(1+\sum_{i=t}^{r}\binom{m-t}{i-t}\right)$. Then we form A from the concatenation of $[K_m^0 K_m^1 \cdots K_m^r]$ and the element-block incidence matrix of an $t-(m, r+1, s)$ packing that has $\Theta(sm^r)$ columns. The result will have $\Theta(m^{t+\alpha})$ columns where the constant will depend on α, t. In particular the constant would shrink with larger α. ∎

We use this result extensively in Section 3 to provide lower bounds. The results given there consider a fixed simple configuration F and then a configuration $F(s)$ obtained from F by repeating certain selected columns s times. The cases considered have already had the asymptotic growth of $\mathrm{forb}(m, F(s))$ determined for fixed s. This paper considers $s = m^\alpha$ times. The following basic result for $F = K_k$ is extended to a result for $F = m^\alpha \cdot K_k$.

Theorem 1.8 *(Sauer[13], Perles and Shelah[14], Vapnik and Chervonenkis[15])*

$$\mathit{forb}(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0}. \qquad (2)$$

Theorem 1.9 *Assume k, α are given and $m^\alpha \ge 2$. Then $\mathit{forb}(m, m^\alpha \cdot K_k)$ is $\Theta(m^{k+\alpha})$.*

Proof: The lower bound follows from Theorem 1.7 since $\mathbf{1}_k \prec K_k$. The upper bound follows using $\mathrm{forb}(m, m^\alpha \cdot K_k) = \mathrm{forb}(m, m^\alpha \cdot \mathbf{1}_k)$ (Theorem 4.4 in [6]). ∎

2 New Induction

We consider a new form of the standard induction [1] for forbidden configurations [2]. For a matrix A, let $\mu(\mathbf{x}, A)$ denote the multiplicity of $\mathbf{x}$ as a column of A. We say A is $(s-1)$-simple if $\mu(\mathbf{x}, A) \le s-1$ for all columns $\mathbf{x}$. We define Let F be a matrix with maximum column multiplicity t. Thus $F \prec t \cdot \text{supp}(F)$. Let $A \in \text{Avoid}(m, F, t-1)$. Assume $\|A\| = \text{forb}(m, \mathcal{F}, t-1)$. Given a row r we permute rows and columns of A to obtain

$$A = \begin{array}{c} \text{row } r \rightarrow \\ \\ \end{array} \begin{bmatrix} 0\,0 \cdots 0 & 1\,1 \cdots 1 \\ G & H \end{bmatrix}. \tag{3}$$

Now $\mu(\alpha, G) \leq t$ and $\mu(\alpha, H) \leq t$. For those α for which $\mu(\alpha, [G\,H]) > t$, let C be formed with $\mu(\alpha, C) = \min\{\mu(\alpha, G), \mu(\alpha, H)\}$. We rewrite our decomposition of A as follows:

$$A = \begin{array}{c} \text{row } r \rightarrow \\ \\ \end{array} \begin{bmatrix} 0\,0 \cdots 0 & 1\,1 \cdots 1 \\ B \quad C & C \quad D \end{bmatrix}. \tag{4}$$

Then we deduce that $[BCD]$ and C are both $(t-1)$-simple. The former follows from $\mu(\alpha, [B\,C\,D]) = \mu(\alpha, G) + \mu(\alpha, H) - \min\{\mu(\alpha, G), \mu(\alpha, H)\} \leq t$. We have that $F \not\prec [B\,C\,D]$ for $F \in \mathcal{F}$. Since each column α of C has $\mu(\alpha, [G\,H]) \geq t$, we deduce that $\text{supp}(F) \not\prec C$ for each $F \in \mathcal{F}$. One induction on m becomes:

$$\begin{aligned} \text{forb}(m, \mathcal{F}, t-1) &= \|A\| = \|[BCD]\| + \|C\| \\ &\leq \text{forb}(m-1, \mathcal{F}, t-1) + (t-1) \cdot \text{forb}(m-1, \{\text{supp}(F) : F \in \mathcal{F}\}). \end{aligned} \tag{5}$$

This is a simplified version of what appears in [1] only using the fact that $C \in \text{Avoid}(m-1, \text{supp}(F))$ and keeping track of the various C's as they are produced.

Lemma 2.1 *Let m, s be given. Let $\mathcal{F}$ consist of simple matrices. Then*

$$\textit{forb}(m, \{s \cdot F : F \in \mathcal{F}\}) \leq \sum_{i=1}^{m} (s-1) \cdot \textit{forb}(m-i, \mathcal{F}). \tag{6}$$

Proof: Apply induction using $\text{forb}(m, s \cdot F) \leq \text{forb}(m, s \cdot F, t)$ so that $\text{forb}(m, \{s \cdot F : F \in \mathcal{F}\}) \leq \text{forb}(m, \{s \cdot F : F \in \mathcal{F}\}, s)$. We keep s is fixed during the course of the induction but s can be a function of m, e.g. $s = m$ is possible. ■

3 Large Configurations

For convenience when dealing with asymptotics we take $s = m^\alpha$. When we say $f(m)$ is $\Theta(m^{2+\alpha})$ for a given α then we are allowing a constant c_α with $f(m) \leq c_\alpha m^{2+\alpha}$.

Theorem 3.1 *Let $\mathbf{1}_k \prec F$ and assume $\textit{forb}(m, F)$ is $\Theta(m^{k-1})$. Then for $2 \leq s$, $\textit{forb}(m, s \cdot F)$ is $\Theta(s \cdot m^k)$. Thus $\textit{forb}(m, m^\alpha \cdot F)$ is $\Theta(m^{k+\alpha})$.*

Proof: The lower bound follows Lemma 1.7. The upper bound follows from (6). Note that if there is a constant c so that for all $i \geq 1$, $\mathrm{forb}(i, F) \leq c \cdot \sum_{i=0}^{k-1} \binom{i}{j}$, then $\sum_{i=1}^{m} \mathrm{forb}(i, F) \leq c \cdot \sum_{i=0}^{k} \binom{m}{j}$. ■

There are many F with the desired property [2]. Consider the following configurations.

$$F_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad F_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad F_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Theorem 3.2 *[2] forb*$(m, F_0) = 2$.

Theorem 3.3 *[3] forb*$(m, F_1) = 4m - 4$.

Theorem 3.4 *[7] forb*(m, F_2) *is* $\Theta(m^2)$.

The following are sample *large* forbidden configuration theorems.

Corollary 3.5 *Let* α *be given. Then forb*$(m, m^\alpha \cdot F_0)$ *is* $O(m^{1+\alpha})$ *and forb*$(m, m^\alpha \cdot F_1)$ *is* $O(m^{2+\alpha})$ *and forb*$(m, m^\alpha \cdot F_2)$ *is* $O(m^{3+\alpha})$.

Proof: The lower bounds arise from Theorem 1.7 and the upper bounds follow by induction from (6) using the bounds of Theorems 3.2,3.3 and 3.4 as appropriate. ■

There are other families of examples. Let I_k denote the $k \times k$ identity matrix, let I_k^c denote the (0,1)-complement of I_k and let T_k denote the upper triangular matrix with the (i, j) entry equal 1 if and only if $i \leq j$. The following is the essential way to obtain a constant bound:

Theorem 3.6 *[8] Let k be given. There is a constant* c_k *with forb*$(m, \{I_k, I_k^c, T_k\}) = c_k$.

If, in the following result, m^α is replaced by a fixed $s \geq 2$, the bound of $\Theta(m)$ was established in [1].

Theorem 3.7 *Let* α *be given with* $m^\alpha \geq 2$. *Then forb*$(m, \{m^\alpha \cdot I_k, m^\alpha \cdot I_k^c, m^\alpha \cdot T_k)$ *is* $\Theta(m^{1+\alpha})$.

Proof: Note that $F_0 \prec I_k$, $F_0 \prec I_k^c$ and $F_0 \prec T_k$. Thus $\mathrm{forb}(m, \{m^\alpha \cdot I_k, m^\alpha \cdot I_k^c, m^\alpha \cdot T_k) \geq \mathrm{forb}(m, m^\alpha \cdot F_0) \geq \mathrm{forb}(m, m^\alpha \cdot \mathbf{1}_1)$. The lower bound follows from Theorem 1.7 when $t = 1$. The upper bound follows from the induction (6) and Theorem 3.6. ■

For a set $S \subseteq [m]$, define $A|_S$ to be the submatrix of A consisting of the rows of S. Let B be a $k \times (k+1)$ simple matrix with one column of each column sum and let $F_B(s) = [K_k \,|\, (s-1) \cdot [K_k \backslash B]]$, where the notation $C \backslash D$ refer to the matrix contained from C by deleting all columns in D. Let $D_{12}(k)$ be the k-rowed simple matrix of all columns which do not have $\mathbf{1}_2$ in rows 1,2 and also does not have the column of 0's. Let $F_{12}(s) = [\mathbf{0}_k \,|\, s \cdot D_{12}(k]$. Note that in either matrix there is a column of multiplicity 1 and not s.

The major result of [4] yields that $F_B(s)$ and $F_{12}(s)$ are the *maximal* k-rowed matrices which have forb being $O(m^{k-1})$. Here maximal means that adding a column not already present s times will result in a configurations with forb being $\Omega(m^k)$. The following lemma is helpful.

Lemma 3.8 *(Lemma 4.4 in [5]) Let A, k and u be given and assume that $\mathcal{S}$ denotes those k-sets of rows S of A for which $A|_S$ has at least two different $k \times 1$ columns α, β with $\mu(\alpha, A|_S) < u$ and $\mu(\beta, A|_S) < u$. We can delete $O(m^{k-1})$ columns from A to obtain A' so that for each $S \in \mathcal{S}$, there is some column not present in $A|_S$.*

In fact there exists a set of at most

$$2u\left(\binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0}\right)$$

columns to delete from A to obtain A'. We use this with $u = m^\alpha$. The following proof is in [4].

Theorem 3.9 *Let k be given and B be a given $k \times (k+1)$ simple matrix with one column of each column sum. Then forb$(m, F_B(m^\alpha))$ is $\Theta(m^{k-1+\alpha})$.*

Proof: The lower bound follows from Theorem 1.7 with k replaced by $k-1$. The upper bound follows from considering $A \in \text{Avoid}(m, F_B(m^\alpha))$ and noting that for each k-set of rows either there is some column not present in $A|_S$ or there are two $k \times 1$ columns α, β with $\mu(\alpha, A|_S) < m^\alpha$ and $\mu(\beta, A|_S) < m^\alpha$. Now by Lemma 3.8 with $u = m^\alpha$, we obtain a matrix A' where for every k-set S there is some column not present in $A|_S$. Thus $A' \in \text{Avoid}(m, K_k)$ and Theorem 1.8 completes the proof. ∎

The proof for $F_{12}(m^\alpha)$ again follows the proof in [5].

Theorem 3.10 *Let k be given. Then forb$(m, F_{12}(m^\alpha))$ is $\Theta(m^{k-1+\alpha})$.*

Proof: The lower bound follows from Theorem 1.7 with k replaced by $k-1$. The upper bound follows from considering $A \in \text{Avoid}(m, F_{12}(m^\alpha))$. Section 2 in [5] outlines the proof. Let $\mathcal{S}$ denotes those k-sets of rows S of A for which $A|_S$ has at least two different $k \times 1$ columns with $\mu(\alpha, A|_S) < m^\alpha$ and $\mu(\beta, A|_S) < m^\alpha$. By Lemma 3.8, one can delete at most $O(m^{k-1+\alpha})$ columns from A to obtain A' where now $\mu(\alpha, A|_S) = \mu(\beta, A|_S) = 0$. We delete a further $O(m^{k-1+\alpha})$ columns from A' to obtain A'' which have no 'violated inner implications' and then deleting a further $O(m^{k-1+\alpha})$ columns from A'' to obtain A''' where there are only a restricted number of 'violated outer implications' and finally deleting a further $O(m^{k-1+\alpha})$ columns from A''' to obtain $A'''' \in \text{Avoid}(m, K_k)$ with no 'violated outer implications' . Then by Theorem 1.8, $\|A''''\|$ is $O(m^{k-1})$. ■

An open problem is to do the same analysis for the following configuration named the 'chestnut'. Define

$$F_3 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

The following bound is implictly given in [11].

Theorem 3.11 *[11] forb*$(m, F_3) = \binom{m}{2} + 2m - 1$

In [1], an induction for the chestnut considers $\text{forb}(m, t \cdot F_3) \le \sum_{i=1}^{m-1} \text{forb}(i, \{F_3, t \cdot [\mathbf{1}_1\mathbf{0}_2\,|\mathbf{0}_1\mathbf{1}_2]\})$ for fixed t. It is shown that $\text{forb}(m, \{F_3, t \cdot [\mathbf{1}_1\mathbf{0}_2\,|\mathbf{0}_1\mathbf{1}_2]\}) \le ctm$ for some constant c. Combining Theorem 3.11 we have that $\text{forb}(m, \{F_3, t \cdot [\mathbf{1}_1\mathbf{0}_2\,|\mathbf{0}_1\mathbf{1}_2]\}) \le \min\{ctm, \binom{m}{2} + 2m - 1\}$. Induction (6) yields $\text{forb}(m, t \cdot F_3) \le tm \cdot \min\{m^2, ctm\}$. For fixed t, this yields $\Theta(m^2)$ for which we have a matching construction. For larger $t = m^\alpha$, the lower bound is $\Omega(m^{2+\alpha})$ by Lemma 1.7 and the upper bound is $O(m^{\min\{2+2\alpha, 3+\alpha\}})$. What is the truth?

Problem 3.12 *Determine forb*$(m, m^\alpha \cdot F_3)$.

Another simple related problem is K_4^2 for which it is known that $\text{forb}(m, K_4^2)$ is $\Theta(m^3)$ and $\text{forb}(m, 2 \cdot K_4^2)$ is $\Theta(m^4)$. What is $\text{forb}(m, m^\alpha \cdot K_4^2)$? The lower bound is only $\Omega(m^{2+\alpha})$ by Lemma 1.7. There are other reasonable problems for large forbidden configurations such as $\text{forb}(m, s \cdot \mathbf{1}_{\sqrt{m}})$. For $s = 1$, the bound is easy.

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